Module 2: Analysis of Stress

2.2.1 PRINCIPAL STRESS IN THREE DIMENSIONS

For the three-dimensional case, for principal stresses it is required that three planes of zero shear stress exist, that these planes are mutually perpendicular, and that on these planes the normal stresses have maximum or minimum values. As discussed earlier, these normal stresses are referred to as principal stresses, usually denoted by σ_1, σ_2 and σ_3. The largest stress is represented by σ_1 and the smallest by σ_3.

Again considering an oblique plane x', the normal stress acting on this plane is given by the Equation (2.25).

$$\sigma_{x'} = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\,(\tau_{xy}\, lm + \tau_{yz}\, mn + \tau_{xz}\, ln) \tag{2.27}$$

The problem here is to determine the extreme or stationary values of $\sigma_{x'}$. To accomplish this, we examine the variation of $\sigma_{x'}$ relative to the direction cosines. As l, m and n are not independent, but connected by $l^2 + m^2 + n^2 = 1$, only l and m may be regarded as independent variables.

Thus,

$$\frac{\partial \sigma_{x'}}{\partial l} = 0, \qquad \frac{\partial \sigma_{x'}}{\partial m} = 0 \tag{2.27a}$$

Differentiating Equation (2.27), in terms of the quantities in Equations (2.22a), (2.22b), (2.22c), we obtain

$$T_x + T_z \frac{\partial n}{\partial l} = 0,$$

$$T_y + T_z \frac{\partial n}{\partial m} = 0, \tag{2.27b}$$

From $n^2 = 1 - l^2 - m^2$, we have

$$\frac{\partial n}{\partial l} = -\frac{l}{n} \quad \text{and} \quad \frac{\partial n}{\partial m} = -\frac{m}{n}$$

Introducing the above into Equation (2.27b), the following relationship between the components of T and n is determined

$$\frac{T_x}{l} = \frac{T_y}{m} = \frac{T_z}{n} \tag{2.27c}$$

These proportionalities indicate that the stress resultant must be parallel to the unit normal and therefore contains no shear component. Therefore from Equations (2.22a), (2.22b), (2.22c) we can write as below denoting the principal stress by σ_P

$$T_x = \sigma_P l \quad T_y = \sigma_P m \quad T_z = \sigma_P n \tag{2.27d}$$

These expressions together with Equations (2.22a), (2.22b), (2.22c) lead to

$$(\sigma_x - \sigma_P)l + \tau_{xy} m + \tau_{xz} n = 0$$

$$\tau_{xy} l + (\sigma_y - \sigma_P) m + \tau_{yz} n = 0 \tag{2.28}$$

$$\tau_{xz} l + \tau_{yz} m + (\sigma_z - \sigma_P) n = 0$$

A non-trivial solution for the direction cosines requires that the characteristic determinant should vanish.

$$\begin{vmatrix} (\sigma_x - \sigma_P) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_y - \sigma_P) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma_P) \end{vmatrix} = 0 \tag{2.29}$$

Expanding (2.29) leads to $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ (2.30)

where $I_1 = \sigma_x + \sigma_y + \sigma_z$ (2.30a)

$$I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2 \tag{2.30b}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} \tag{2.30c}$$

The three roots of Equation (2.30) are the principal stresses, corresponding to which are three sets of direction cosines that establish the relationship of the principal planes to the origin of the non-principal axes.

2.2.2 STRESS INVARIANTS

Invariants mean those quantities that are unexchangeable and do not vary under different conditions. In the context of stress tensor, invariants are such quantities that do not change with rotation of axes or which remain unaffected under transformation, from one set of axes to another. Therefore, the combination of stresses at a point that do not change with the orientation of co-ordinate axes is called stress-invariants. Hence, from Equation (2.30)

$\sigma_x + \sigma_y + \sigma_z = I_1$ = First invariant of stress

$\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau^2_{xy} - \tau^2_{yz} - \tau^2_{zx} = I_2$ = Second invariant of stress

$\sigma_x\sigma_y\sigma_z - \sigma_x\tau^2_{yz} - \sigma_y\tau^2_{xz} - \sigma_z\tau^2_{xy} + 2\tau_{xy}\,\tau_{yz}\,\tau_{xz} = I_3$ = Third invariant of stress

2.2.3 EQUILIBRIUM OF A DIFFERENTIAL ELEMENT

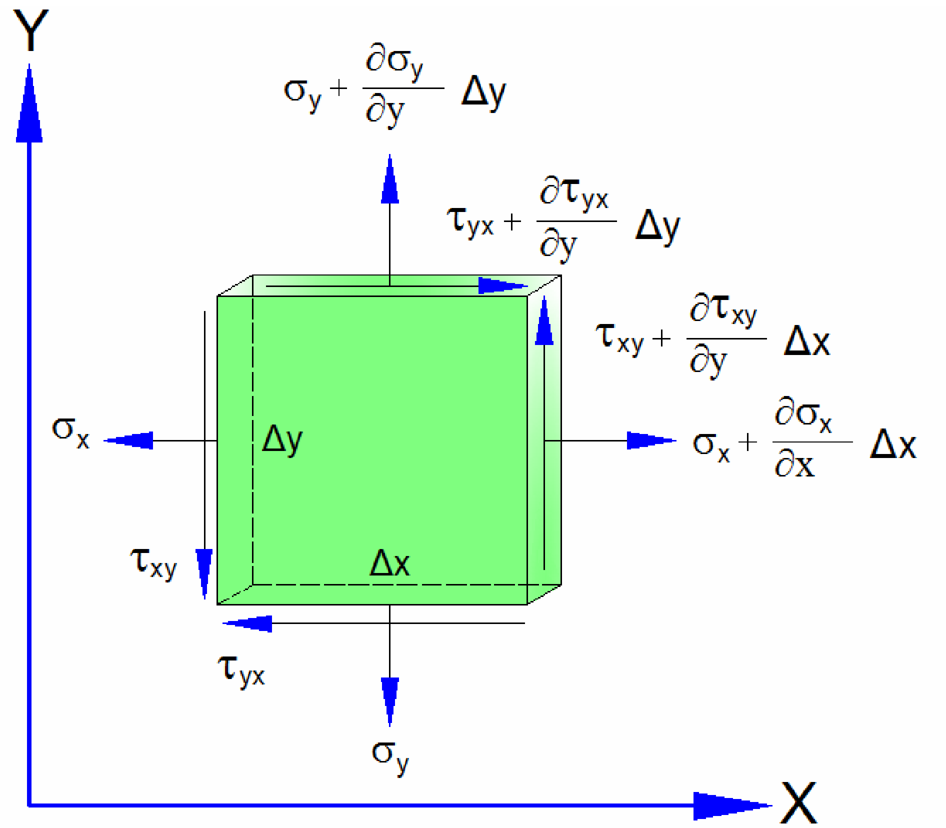


Figure 2.11(a) Stress components acting on a plane element

When a body is in equilibrium, any isolated part of the body is acted upon by an equilibrium set of forces. The small element with unit thickness shown in Figure 2.11(a) represents part of a body and therefore must be in equilibrium if the entire body is to be in equilibrium. It is to be noted that the components of stress generally vary from point to point in a stressed body. These variations are governed by the conditions of equilibrium of statics. Fulfillment of these conditions establishes certain relationships, known as the differential equations of equilibrium. These involve the derivatives of the stress components.

Assume that σ_x, σ_y, τ_{xy}, τ_{yx} are functions of X, Y but do not vary throughout the thickness (are independent of Z) and that the other stress components are zero.

Also assume that the X and Y components of the body forces per unit volume, F_x and F_y, are independent of Z, and that the Z component of the body force $F_z = 0$. As the element is very small, the stress components may be considered to be distributed uniformly over each face.

Now, taking moments of force about the lower left corner and equating to zero,

$$-(\sigma_x \Delta y)\frac{\Delta y}{2} + (\tau_{xy}\Delta y)\frac{1}{2} - \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}\Delta y\right)\Delta x\frac{\Delta x}{2} + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x\Delta y$$

$$-\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x\right)\Delta x\Delta y + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y\frac{\Delta y}{2} + \sigma_y\Delta x\frac{\Delta x}{2} - \tau_{yx}\Delta x\frac{1}{2} +$$

$$(F_x\Delta y\Delta x)\frac{\Delta y}{2} - (F_y\Delta x\Delta y)\frac{\Delta x}{2} = 0$$

Neglecting the higher terms involving Δx, and Δy and simplifying, the above expression is reduced to

$\tau_{xy}\,\Delta x\,\Delta y = \tau_{yx}\,\Delta x\,\Delta y$

or $\quad \tau_{xy} = \tau_{yx}$

In a like manner, it may be shown that

$\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$

Now, from the equilibrium of forces in x-direction, we obtain

$$-\sigma_x\,\Delta y + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x - \tau_{yx}\Delta x + F_x\Delta x\Delta y = 0$$

Simplifying, we get

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0$$

or

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

A similar expression is written to describe the equilibrium of y forces. The x and y equations yield the following differential equations of equilibrium.

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + F_x = 0$$

or $$\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + F_y = 0 \qquad (2.31)$$

The differential equations of equilibrium for the case of three-dimensional stress may be generalized from the above expressions as follows [Figure 2.11(b)].

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + F_x = 0$$
$$\frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yz}}{\partial z} + F_y = 0 \qquad (2.32)$$
$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + F_z = 0$$

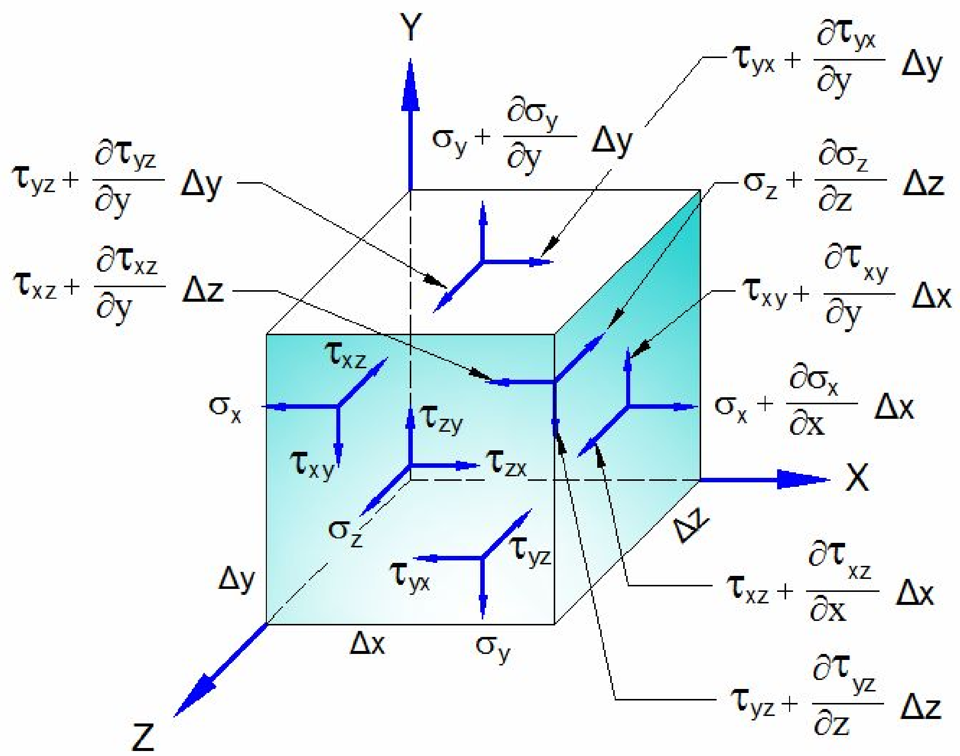
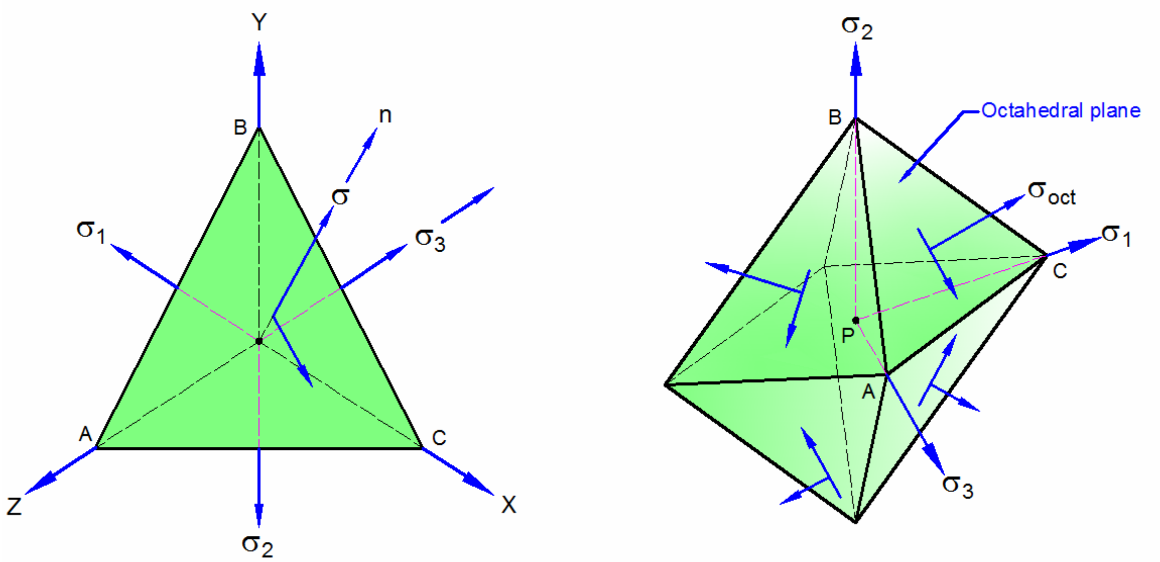


Figure 2.11(b) Stress components acting on a three dimensional element

2.2.4 OCTAHEDRAL STRESSES

A plane which is equally inclined to the three axes of reference, is called the octahedral plane and its direction cosines are $\pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}},\ \pm\frac{1}{\sqrt{3}}$. The normal and shearing stresses acting on this plane are called the octahedral normal stress and octahedral shearing stress respectively. In the Figure 2.12, X, Y, Z axes are parallel to the principal axes and the octahedral planes are defined with respect to the principal axes and not with reference to an arbitrary frame of reference.



(a) **(b)**

Figure 2.12 Octahedral plane and Octahedral stresses

Now, denoting the direction cosines of the plane ABC by l, m, and n, the equations (2.22a), (2.22b) and (2.22c) with $\sigma_x = \sigma_1,\ \tau_{xy} = \tau_{xz} = 0$ etc. reduce to

$$T_x = \sigma_1\ l,\ \ T_y = \sigma_2\ m \text{ and } T_z = \sigma_3\ n \tag{2.33}$$

The resultant stress on the oblique plane is thus

$$T^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 = \sigma^2 + \tau^2$$

$$\therefore\ T^2 = \sigma^2 + \tau^2 \tag{2.34}$$

The normal stress on this plane is given by

$$\sigma = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 \tag{2.35}$$

and the corresponding shear stress is

$$\tau = \left[(\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_2 - \sigma_3)^2 m^2 n^2 + (\sigma_3 - \sigma_1)^2 n^2 l^2\right]^{\frac{1}{2}} \tag{2.36}$$

The direction cosines of the octahedral plane are:

$$l = \pm \frac{1}{\sqrt{3}}, \quad m = \pm \frac{1}{\sqrt{3}}, \quad n = \pm \frac{1}{\sqrt{3}}$$

Substituting in (2.34), (2.35), (2.36), we get

$$\text{Resultant stress } T = \sqrt{\frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)} \tag{2.37}$$

$$\text{Normal stress} = \sigma = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.38}$$

$$\text{Shear stress} = \tau = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{2.39}$$

$$\text{Also, } \tau = \frac{1}{3}\sqrt{2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)} \tag{2.40}$$

$$\tau = \frac{1}{3}\sqrt{2I_1^2 - 6I_2} \tag{2.41}$$

2.2.5 MOHR'S STRESS CIRCLE

A graphical means of representing the stress relationships was discovered by Culmann (1866) and developed in detail by Mohr (1882), after whom the graphical method is now named.

2.2.6 MOHR CIRCLES FOR TWO DIMENSIONAL STRESS SYSTEMS

Biaxial Compression (Figure 2.13a)

The biaxial stresses are represented by a circle that plots in positive (σ, τ) space, passing through stress points σ_1, σ_2 on the $\tau = 0$ axis. The centre of the circle is located on the

$\tau = 0$ axis at stress point $\frac{1}{2}(\sigma_1 + \sigma_2)$. The radius of the circle has the magnitude $\frac{1}{2}(\sigma_1 - \sigma_2)$, which is equal to τ_{max}.

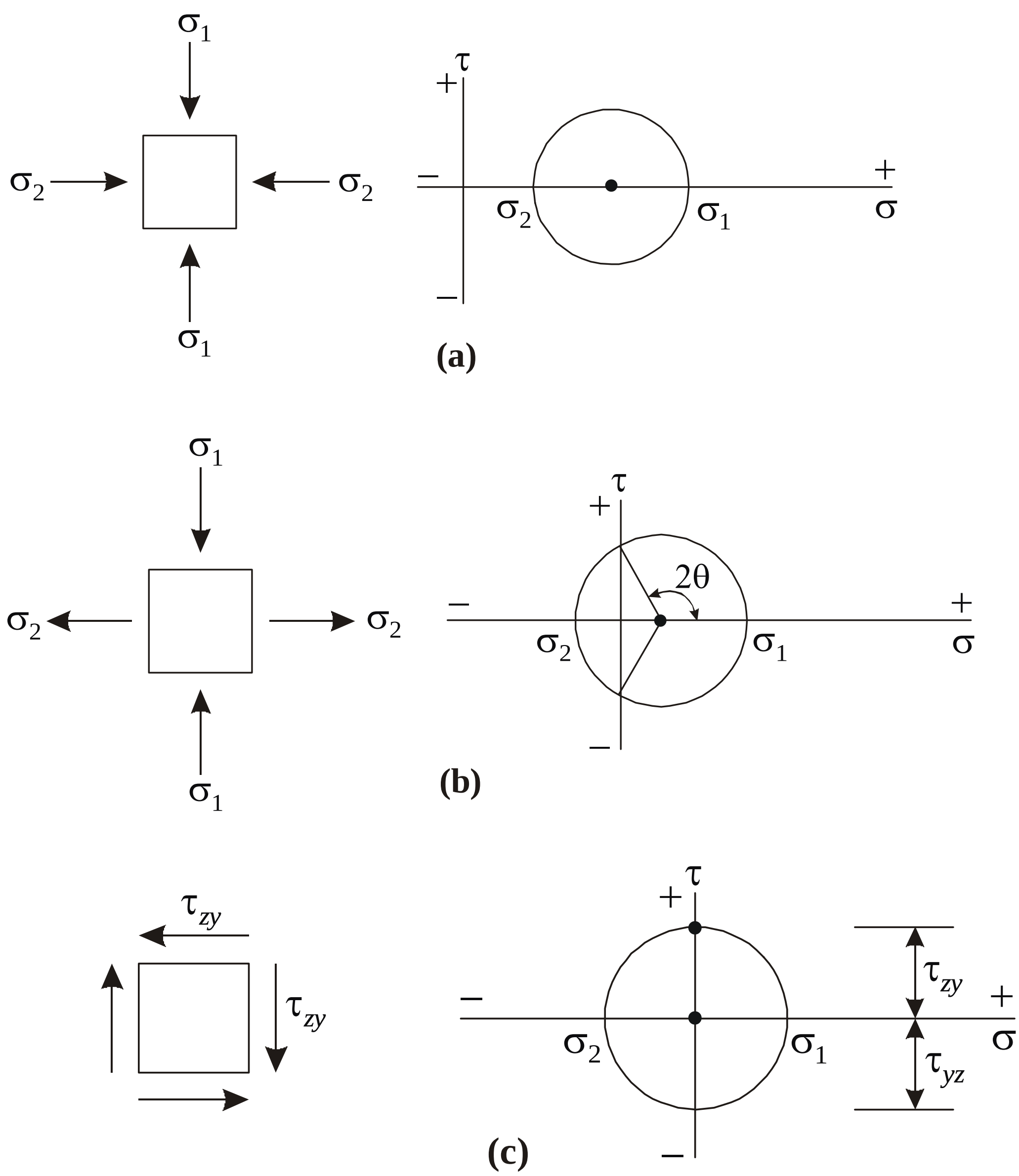


Figure 2.13 Simple Biaxial stress systems: (a) compression, (b) tension/compression, (c) pure shear

Biaxial Compression/Tension (Figure 2.13b)

Here the stress circle extends into both positive and negative σ space. The centre of the circle is located on the $\tau = 0$ axis at stress point $\frac{1}{2}(\sigma_1 + \sigma_2)$ and has radius $\frac{1}{2}(\sigma_1 - \sigma_2)$. This is also the maximum value of shear stress, which occurs in a direction at 45° to the σ_1 direction. The normal stress is zero in directions $\pm\theta$ to the direction of σ_1, where

$$\cos 2\theta = -\frac{\sigma_1 + \sigma_2}{\sigma_1 - \sigma_2}$$

Biaxial Pure Shear (Figure 2.13c)

Here the circle has a radius equal to τ_{zy}, which is equal in magnitude to τ_{yz}, but opposite in sign. The centre of circle is at $\sigma = 0$, $\tau = 0$. The principal stresses σ_1, σ_2 are equal in magnitude, but opposite in sign, and are equal in magnitude to τ_{zy}. The directions of σ_1, σ_2 are at 45° to the directions of τ_{zy}, τ_{yz}

2.2.7 CONSTRUCTION OF MOHR'S CIRCLE FOR TWO-DIMENSIONAL STRESS

Sign Convention

For the purposes of constructing and reading values of stress from Mohr's circle, the sign convention for shear stress is as follows.

If the shearing stresses on opposite faces of an element would produce shearing forces that result in a clockwise couple, these stresses are regarded as "positive".

Procedure for Obtaining Mohr's Circle

1) Establish a rectangular co-ordinate system, indicating $+\tau$ and $+\sigma$. Both stress scales must be identical.
2) Locate the centre C of the circle on the horizontal axis a distance $\frac{1}{2}(\sigma_X + \sigma_Y)$ from the origin as shown in the figure above.
3) Locate point A by co-ordinates $\sigma_x, -\tau_{xy}$
4) Locate the point B by co-ordinates σ_y, τ_{xy}
5) Draw a circle with centre C and of radius equal to CA.
6) Draw a line AB through C.

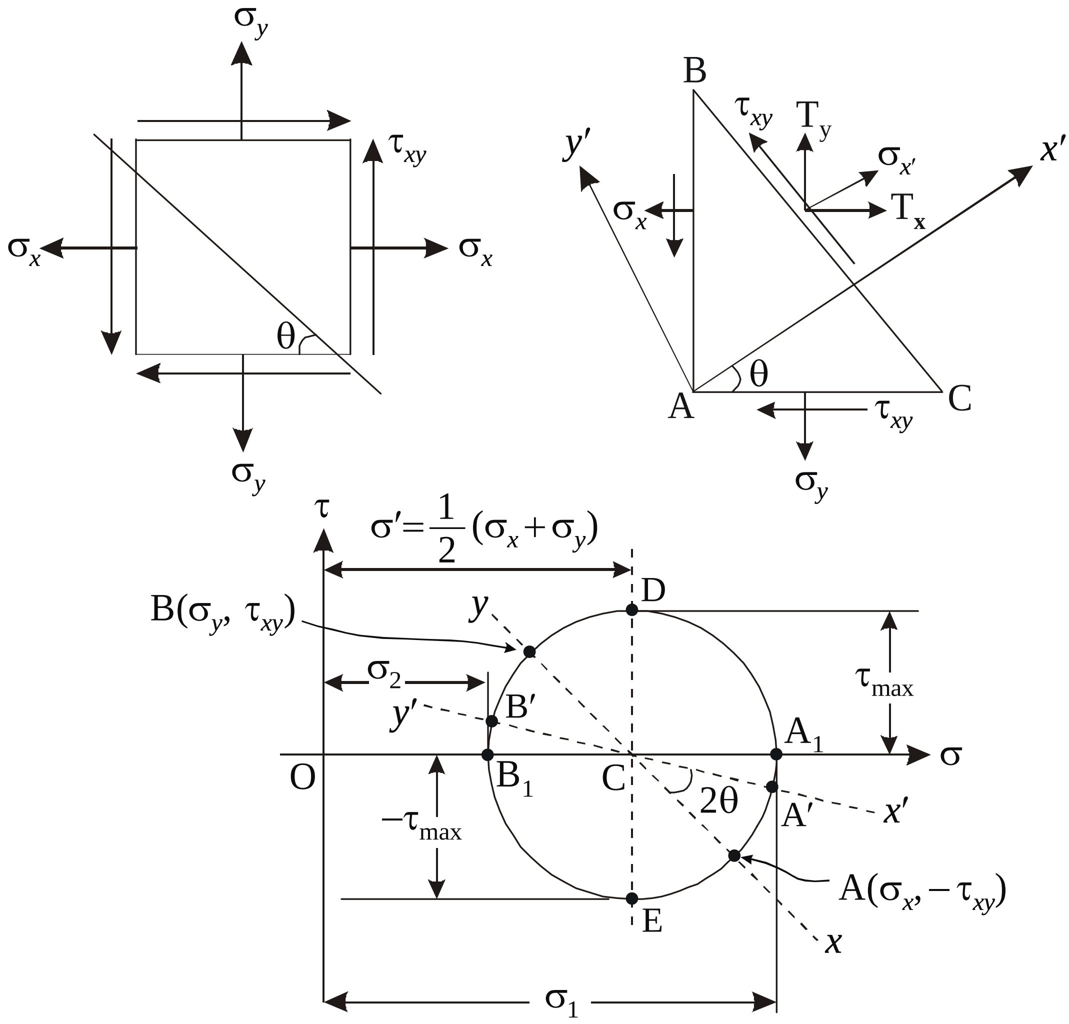


Figure 2.14 Construction of Mohr's circle

An angle of 2θ on the circle corresponds to an angle of θ on the element. The state of stress associated with the original x and y planes corresponds to points A and B on the circle respectively. Points lying on the diameter other than AB, such as A' and B', define state of stress with respect to any other set of x' and y' planes rotated relative to the original set through an angle θ.

It is clear from the figure that the points A_1 and B_1 on the circle locate the principal stresses and provide their magnitudes as defined by Equations (2.14) and (2.15), while D and E represent the maximum shearing stresses. The maximum value of shear stress (regardless of algebraic sign) will be denoted by τ_{max} and are given by

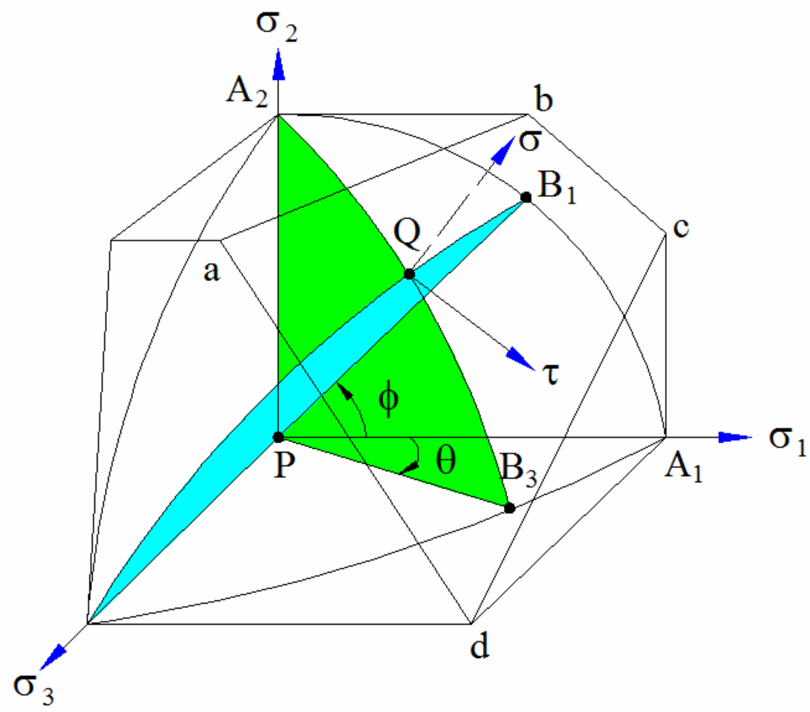
$$\tau_{\max} = \pm\frac{1}{2}(\sigma_1 - \sigma_2) = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (2.42)$$

Mohr's circle shows that the planes of maximum shear are always located at 45° from planes of principal stress.

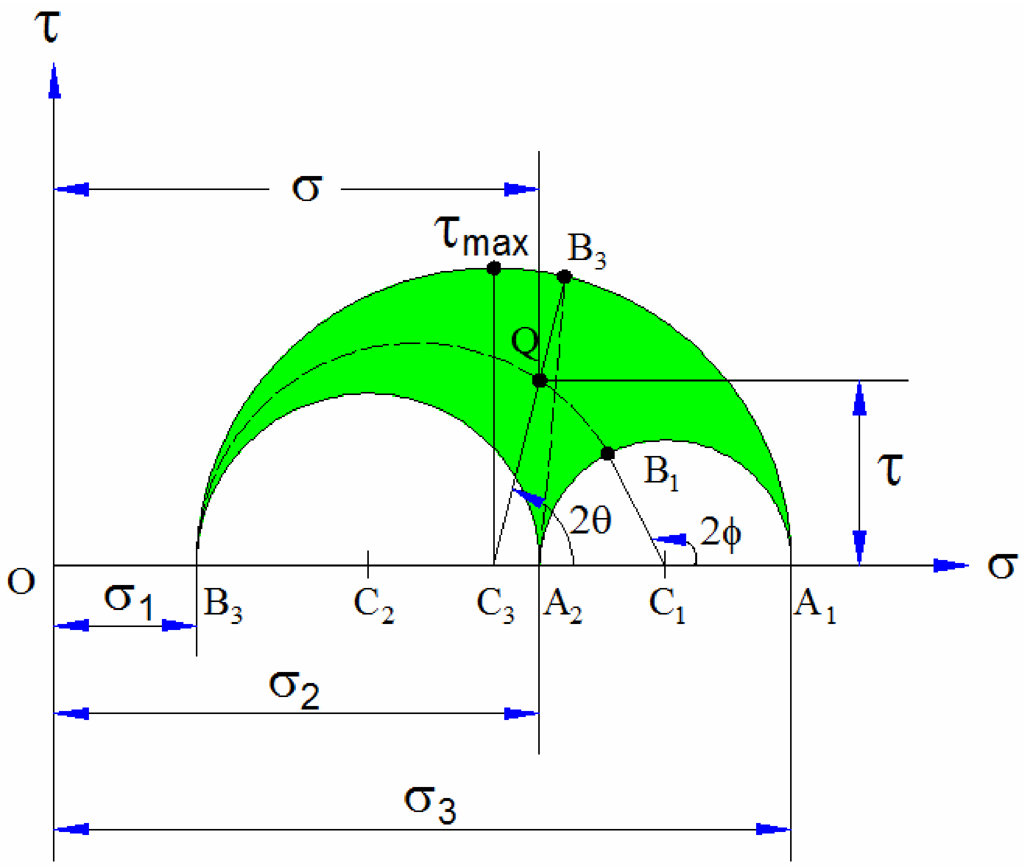
2.2.8 MOHR'S CIRCLE FOR THREE-DIMENSIONAL STATE OF STRESS

When the magnitudes and direction cosines of the principal stresses are given, then the stresses on any oblique plane may be ascertained through the application of Equations (2.33) and (2.34). This may also be accomplished by means of Mohr's circle method, in which the equations are represented by three circles of stress.

Consider an element as shown in the Figure 2.15, resulting from the cutting of a small cube by an oblique plane.



(a)



(b)

Figure 2.15 Mohr's circle for Three Dimensional State of Stress

The element is subjected to principal stresses σ_1, σ_2 and σ_3 represented as coordinate axes with the origin at P. It is required to determine the normal and shear stresses acting at point Q on the slant face (plane abcd). This plane is oriented so as to be tangent at Q to a quadrant of a spherical surface inscribed within a cubic element as shown. It is to be noted that PQ, running from the origin of the principal axis system to point Q, is the line of intersection of the shaded planes (Figure 2.15 (a)). The inclination of plane PA_2QB_3 relative to the σ_1 axis is given by the angle θ (measured in the σ_1, σ_3 plane), and that of plane PA_3QB_1, by the angle Φ (measured in the σ_1 and σ_2 plane). Circular arcs $A_1B_1A_2$ and $A_1B_3A_3$ are located on the cube faces. It is clear that angles θ and Φ unambiguously define the orientation of PQ with respect to the principal axes.

Procedure to determine Normal Stress *(σ)* and Shear Stress *(τ)*

1) Establish a Cartesian co-ordinate system, indicating $+\sigma$ and $+\tau$ as shown. Lay off the principal stresses along the σ-axis, with $\sigma_1 > \sigma_2 > \sigma_3$ (algebraically).

2) Draw three Mohr semicircles centered at C_1, C_2 and C_3 with diameters A_1A_2, A_2A_3 and A_1A_3.

3) At point C_1, draw line $C_1 B_1$ at angle 2ϕ; at C_3, draw $C_3 B_3$ at angle 2θ. These lines cut circles C_1 and C_3 at B_1 and B_3 respectively.

4) By trial and error, draw arcs through points A_3 and B_1 and through A_2 and B_3, with their centres on the σ-axis. The intersection of these arcs locates point Q on the σ, τ plane.

In connection with the construction of Mohr's circle the following points are of particular interest:

a) Point Q will be located within the shaded area or along the circumference of circles C_1, C_2 or C_3, for all combinations of θ and ϕ.

b) For particular case $\theta = \phi = 0$, Q coincides with A_1.

c) When $\theta = 45^0$, $\phi = 0$, the shearing stress is maximum, located as the highest point on circle C_3 ($2\theta = 90^0$). The value of the maximum shearing stress is therefore $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$ acting on the planes bisecting the planes of maximum and minimum principal stresses.

d) When $\theta = \phi = 45^0$, line PQ makes equal angles with the principal axes. The oblique plane is, in this case, an octahedral plane, and the stresses along on the plane, the octahedral stresses.

2.2.9 GENERAL EQUATIONS IN CYLINDRICAL CO-ORDINATES

While discussing the problems with circular boundaries, it is more convenient to use the cylindrical co-ordinates, r, θ, z. In the case of plane-stress or plane-strain problems, we have $\tau_{rz} = \tau_{\theta z} = 0$ and the other stress components are functions of r and θ only. Hence the cylindrical co-ordinates reduce to polar co-ordinates in this case. In general, polar co-ordinates are used advantageously where a degree of axial symmetry exists. Examples include a cylinder, a disk, a curved beam, and a large thin plate containing a circular hole.

2.2.10 EQUILIBRIUM EQUATIONS IN POLAR CO-ORDINATES: (TWO-DIMENSIONAL STATE OF STRESS)

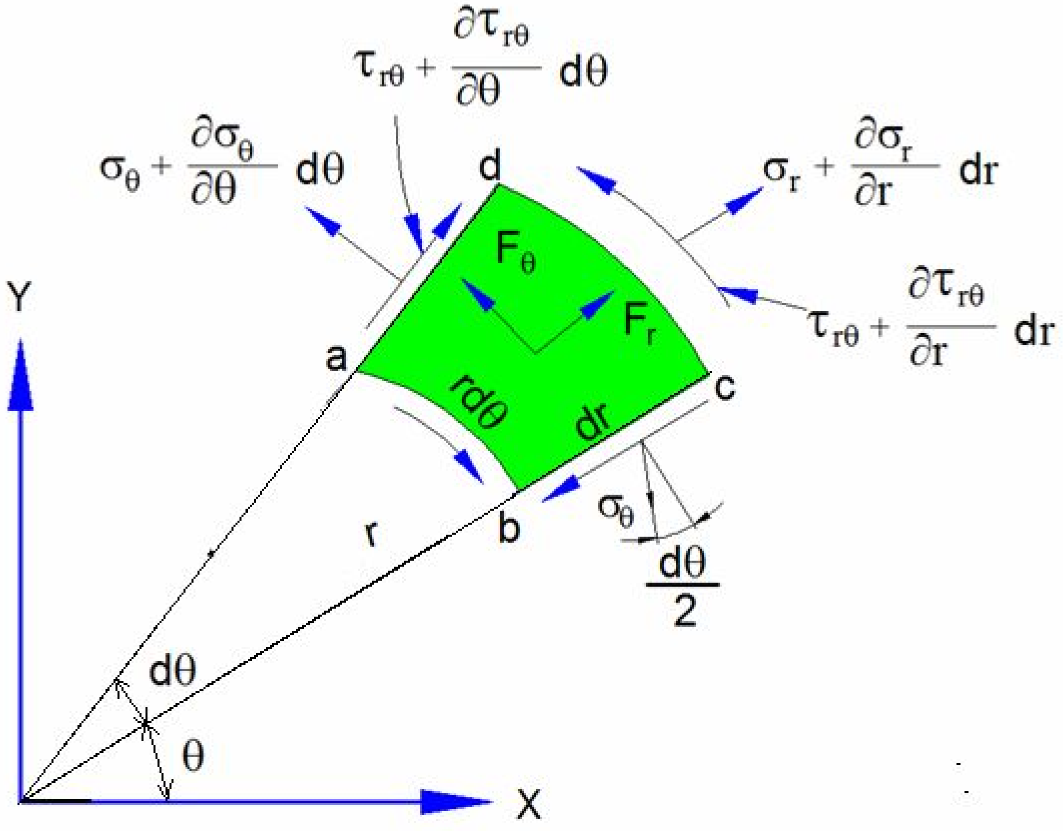


Figure 2.16 Stresses acting on an element

The polar coordinate system (r, θ) and the cartesian system (x, y) are related by the following expressions:

$$x = r\cos\theta, \qquad r^2 = x^2+y^2$$

$$y = r\sin\theta, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{2.43}$$

Consider the state of stress on an infinitesimal element abcd of unit thickness described by the polar coordinates as shown in the Figure 2.16. The body forces denoted by F_r and F_θ are directed along r and θ directions respectively.

Resolving the forces in the r-direction, we have for equilibrium, $\Sigma F_r = 0$,

$$-\sigma_r \times rd\theta + \left(\sigma_r + \frac{\partial\sigma_r}{\partial r}dr\right)(r+dr)d\theta - \sigma_\theta dr\sin\frac{d\theta}{2} + F_r - \left(\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta}d\theta\right)dr$$

$$\sin\frac{d\theta}{2} - \tau_{r\theta}dr\cos\frac{d\theta}{2} + \left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta}d\theta\right)dr\cos\frac{d\theta}{2} = 0$$

Since $d\theta$ is very small,

$$\sin\frac{d\theta}{2} = \frac{d\theta}{2} \text{ and } \cos\frac{d\theta}{2} = 1$$

Neglecting higher order terms and simplifying, we get

$$r\frac{\partial\sigma_r}{\partial r}dr\,d\theta + \sigma_r dr\,d\theta - \sigma_\theta dr\,d\theta + \frac{\partial\tau_{r\theta}}{\partial\theta}dr\,d\theta = 0$$

on dividing throughout by $rd\theta\ dr$, we have

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \qquad (2.44)$$

Similarly resolving all the forces in the θ - direction at right angles to r - direction, we have

$$-\sigma_\theta dr\cos\frac{d\theta}{2} + \left(\sigma_\theta + \frac{\partial\sigma_\theta}{\partial\theta}d\theta\right)dr\cos\frac{d\theta}{2} + \tau_{r\theta}dr\sin\frac{d\theta}{2} + \left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial\theta}d\theta\right)dr$$

$$\sin\frac{d\theta}{2} - \tau_{r\theta}rd\theta + (r+dr)d\theta\left(\tau_{r\theta} + \frac{\partial\tau_{r\theta}}{\partial r}dr\right) + F_\theta = 0$$

On simplification, we get

$$\left(\frac{\partial\sigma_\theta}{\partial\theta} + \tau_{r\theta} + \tau_{r\theta} + r\frac{\partial\tau_{r\theta}}{\partial r}\right)d\theta\,dr = 0$$

Dividing throughout by $rd\theta\,dr$, we get

$$\frac{1}{r}\cdot\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0 \qquad (2.45)$$

In the absence of body forces, the equilibrium equations can be represented as:

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (2.46)$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$